Table Entries: Derivative Rules

1. t-derivative rule

This is a course on differential equations. We should try to compute $\mathcal{L}(f')$. (We use the notation f' instead of $\dot{f}$ simply because we think the dot does not sit nicely over the tall letter f.)

As usual, let $\mathcal{L}(f)(s) = F(s)$. Let f' be the *generalized* derivative of f. (Recall, this means jumps in f produce delta functions in f'.) The t-derivative rule is

$$\begin{aligned}
\mathcal{L}(f') &= sF(s) - f(0^-) &\quad (1)\\
\mathcal{L}(f'') &= s^2F(s) - sf(0^-) - f'(0^-) &\quad (2)\\
\mathcal{L}(f^{(n)}) &= s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) + \ldots + f^{(n-1)}(0^-). &\quad (3)
\end{aligned}$$

Proof: Rule (1) is a simple consequence of the definition of Laplace transform and integration by parts.

$$\begin{aligned}
\mathcal{L}(f') &= \int_{0^-}^{\infty} f'(t)e^{-st}\,dt & & u = e^{-st} & v' = f'(t)\\
& & & u' = -se^{-st} & v = f(t)\\
&= f(t)e^{-st}\Big]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt\\
&= -f(0^-) + sF(s).
\end{aligned}$$

The last equality follows from:
1. We assume $f(t)$ has exponential order, so if $\text{Re}(s)$ is large enough $f(t)e^{-st}$ is 0 at $t = \infty$.
2. The integral in the second term is none other than the Laplace transform of $f(t)$.

Rule (2) follows by applying rule (1) twice.

$$\begin{aligned}
\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0^-)\\
&= s(\mathcal{L}(f) - f(0^-)) - f'(0^-)\\
&= sF(s) - sf(0^-) - f'(0^-).
\end{aligned}$$

Rule (3) Follows by applying rule (1) n times.

Notes: 1. We will call the terms $f(0^-)$, $f'(0^-)$ the 'annoying terms'. We will be happiest when our signal $f(t)$ has rest initial conditions, so all of

the annoying terms are 0.
2. A good way to think of the t-derivative rules is

$$\begin{aligned}\mathcal{L}(f) &= F(s)\\ \mathcal{L}(f') &= sF(s) + \text{ annoying terms at } 0^-.\\ \mathcal{L}(f'') &= s^2F(s) + \text{ annoying terms at } 0^-.\end{aligned}$$

Roughly speaking, Laplace transforms differentiation in t to multiplication by s.
3. The proof of rule (1) uses integration by parts. This is clearly valid if $f'(t)$ is continuous at $t = 0$. It is also true (although we won't show this) if $f'(t)$ is a generalized function. –See example 2 below.

Example 1. Let $f(t) = e^{at}$. We can compute $\mathcal{L}(f')$ directly and by using rule (1).
Directly: $f'(t) = ae^{at} \Rightarrow \mathcal{L}(f') = a/(s-a)$.
Rule (1): $\mathcal{L}(f) = F(s) = 1/(s-a) \Rightarrow \mathcal{L}(f') = sF(s) - f(0^-) = s/(s-a) - 1 = a/(s-a)$.
Both methods give the same answer.

Example 2. Let $u(t)$ be the unit step function, so $\dot{u}(t) = \delta(t)$.
Directly: $\mathcal{L}(\dot{u}) = \mathcal{L}(\delta) = 1$.
Rule (1): $\mathcal{L}(\dot{u}) = s\mathcal{L}(u) - u(0^-) = s(1/s) - 0 = 1$.
Both methods give the same answer.

Example 3. Let $f(t) = t^2 + 2t + 1$. Compute $\mathcal{L}(f'')$ two ways.

Solution. Directly: $f''(t) = 2 \Rightarrow \mathcal{L}(f'') = 2/s$.
Using rule (3): $\mathcal{L}(f'') = s^2F(s) - sf(0^-) - f'(0^-) = s^2(2/s^3 + 2/s^2 + 1/s) - s \cdot 1 - 2 = 2/s$.
Both methods give the same answer.

2. s-derivative rule

There is a certain symmetry in our formulas. If derivatives in time lead to multiplication by s then multiplication by t should lead to derivatives in s. This is true, but, as usual, there are small differences in the details of the formulas.

The s-derivative rule is

$$\begin{aligned}\mathcal{L}(tf)(s) &= -F'(s) && (4)\\ \mathcal{L}(t^nf)(s) &= (-1)^nF^{(n)}(s) && (5)\\ & && (6)\end{aligned}$$

Proof: Rule (4) is a simple consequence of the definition of Laplace transform.

$$\begin{aligned} F(s) &= \mathcal{L}(f) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \\ \Rightarrow F'(s) &= \frac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} -tf(t)e^{-st} \\ &= -\mathcal{L}(tf(t)). \end{aligned}$$

Rule (5) is just rule (4) applied n times.

Example 4. Use the s-derivative rule to find $\mathcal{L}(t)$.

Solution. Start with $f(t) = 1$, then $F(s) = 1/s$. The s-derivative rule now says $\mathcal{L}(t) = -F'(s) = 1/s^2$ –which we know to be the answer.

Example 5. Use the s-derivative rule to find $\mathcal{L}(te^{at}$ and $\mathcal{L}(t^n e^{at})$.

Solution. Start with $f(t) = e^{at}$, then $F(s) = 1/(s-a)$. The s-derivative rule now says $\mathcal{L}(te^{at}) = -F'(s) = 1/(s-a)^2$.

Continuing: $\mathcal{L}(t^2e^{at}) = F''(s) = 2/(s-a)^3$,
$\mathcal{L}(t^3e^{at}) = -F'''(s) = 3\cdot 2/(s-a)^4$, $\mathcal{L}(t^4e^{at}) = F^{(4)}(s) = 4\cdot 3\cdot 2/(s-a)^5$,
$\mathcal{L}(t^ne^{at}) = (-1)^nF^{(n)}(s) = n!/(s-a)^{n+1}$.

With Laplace, there is often more than one way to compute. We know $\mathcal{L}(t^n) = n!/s^{n+1}$. Therefore the s-shift rule also gives the above formula for $\mathcal{L}(t^ne^{at})$.

3. Repeated Quadratic Factors

Recall the table entries for repeated quadratic factors

$$\mathcal{L}\left(\frac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))\right) = \frac{1}{(s^2+\omega^2)^2} \tag{7}$$

$$\mathcal{L}\left(\frac{t}{2\omega}\sin(\omega t)\right) = \frac{s}{(s^2+\omega^2)^2} \tag{8}$$

$$\mathcal{L}\left(\frac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))\right) = \frac{s^2}{(s^2+\omega^2)^2} \tag{9}$$

Previously we proved thesse formulas using partial fractions and factoring the denominators on the frequency side into complex linear factors. Let's prove them again using the s-derivative rule.

Proof of (8) using the s-derivative rule.
Let $f(t) = \sin(\omega t)$. We know $F(s) = \dfrac{\omega}{s^2 + \omega^2}$. The s-derivative rule implies

$$\mathcal{L}(t \sin \omega t) = -F'(s) = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

This formula is (8) with the factor of 2ω moved from one side to the other.

The other two formulas can be proved in a similar fashion. We won't give the proofs here.

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